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LETTER TO THE EDITOR

An investigation of fractal dimensions in two-dimensional lattice gas turbulence

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Abstract. We investigate the dissipation mechanisms acting in a two-dimensional lattice gas automaton by inspecting the structure functions of the turbulent velocity field associated with the Boolean configuration of the automaton. In particular, we investigate whether the Boolean noise produced by the automaton can promote fractal structures within the flow. We show that this is not the case and the presence of the noise only results in a non-analyticity of the flow field which can be progressively eliminated upon averaging the boolean field on coarser and coarser grids. As a result, we find that the non-fractal nature of homogeneous two-dimensional turbulence is not affected by the presence of the microscopic noise.

It has been pointed out recently that lattice gas automata provide a valuable tool to simulate the Navier-Stokes equation and to study the influence of 'noisy hydrodynamics' on a turbulent flow. In a lattice gas 'noisy hydrodynamics' is intrinsically related to the fluctuations of the 'Boolean molecules' and consequently a detailed study of the dissipation mechanisms can be carried out.

In this letter we address this topic with explicit reference to a two-dimensional turbulent flow. To this aim, we have simulated the two-dimensional Navier-Stokes equation with a 8192^2 FHP-III lattice gas automaton [1, 2]. The initial boolean field is a stochastic realisation of a 512^2 spectral simulation of 2D decaying turbulence at $t=30$ [3]. The simulation took about 70 megabytes of storage and 50 h CPU time on a single processor of the IBM 3090/200 with vector facility [4]; further details can be found in [5] where a quantitative comparison is made between spectral and lattice gas simulation.

In a fluid flow the dissipation is governed by the gradient of the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$. Consequently, the quantities

$$A_p(\chi, \eta) = \langle |v(x+\chi, y+\eta) - v(x, y)|^p \rangle \tag{1}$$

provide a quantitative indication on the way energy and enstrophy are transferred from large to short scales and there dissipated. The functions A_p, usually referred to as structure functions, have been introduced to study the fractal and multifractal nature of turbulent flows [6]. For small values of the displacement $\boldsymbol{r} \equiv (\chi, \eta)$ the quantity A_p scales like $r^{\alpha(p)}$ and the behaviour of the coefficient $\alpha(p)$ as a function of p provides a quantitative indication on the singularities of the flow field. For example, a linear law $\alpha(p)=p$ indicates that the velocity field is differentiable at least once so that its gradient is not singular.

It has been suggested that in a 3D turbulent flow $\alpha_0 \equiv \alpha(p \to 0) = 3 - D_F$ where D_F is the fractal dimension of the set where the gradient of v is singular. On the other hand, for a 2D turbulent flow one can easily prove that $\alpha(p) = p$ because of the following constraints [7]:

(i) α is a convex function of p;

(ii) $\alpha(3) = 3$;

(iii) $\alpha(p) \geq p$.

Constraint (iii) stems from the inequality $A_1(r) \leq Cr \ln r$ which has been proved analytically for the Euler equation [8] and can be readily extended to the viscous case. Since $\alpha(p) = p$ the gradient of v is regular everywhere and dissipation is a smooth and homogeneous process all over the fluid flow. This amounts to saying that there is no intermittency of the velocity field at very small scales. As a result, there is a tendency to form coherent structures which dissipate vorticity at a very small rate [3]. In a lattice gas this need not be true because the Navier–Stokes equation does not include the 'molecular' noise which is naturally inherent to the Boolean gas. We argue that if the 'molecular noise' tends to generate fractal structures at the dissipation scale, large-scale coherent vortices might dissipate their vorticity at much higher rates than those pertaining to the 'noiseless' Navier–Stokes picture. Should this be the case, the time evolution of a large-scale field would be significantly affected by the noise.

The quantities in (1) were obtained as spatial averages taken over the computational domain, a square box of size 2π, along various directions of the displacement vector $\boldsymbol{r}$. We have computed $\alpha(p)$ both for the velocity field obtained by a direct integration of the Navier–Stokes equations and for the velocity field obtained by the lattice gas simulation.

In the latter case, the macroscopic velocity field is obtained by mapping the 8192^2 Boolean configuration onto a 512^2 flow field in such a way that each velocity is made up of 16^2 bits. The Boolean configurations refer to the evolution over 4×8192 time steps of the automaton which correspond to about 1.2 spectral time units. The structure function obtained by the spectral simulation is reported in the curve A of figure 1, while curves B and C refer to the velocity field obtained by the lattice gas simulation mapped onto a 512^2 and a 128^2 grid respectively.

For curves A and B the function α displays a linear behaviour $\alpha = hp$, although with distinct values of the slope h. For the spectral field (curve A) we find $h \sim 1$, as it must be according to the regularity constraints mentioned above, while for the lattice gas field (curve B) the coefficient h is of the order of 0.4. The picture emerging from these data seems quite clear and transparent except for the non-linear behaviour of curve C for p slightly above 1 which is somehow unexpected. As a first reaction, one is led to relate this non-linear behaviour to a lack of statistics which calls for further analysis. Along this line, we consider the statistical dispersion introduced by computing the spatial averages in (1) as discrete sums taken over a finite set of grid points. However, because these sums involve 128^2 points, the corresponding statistical dispersion is of the order of $1/128 \sim 0.0078$ which is smaller than the deviation of curve C from the straight line followed at $p < 1$. A number of further refinements, such as a careful choice of the range of values of r used to evaluate the ratio $\ln A_p / \ln r$, were found to yield a statistical variance always well below 0.1 which is again too small for a linear behaviour to be recovered. As a result, we conclude that the possibility that

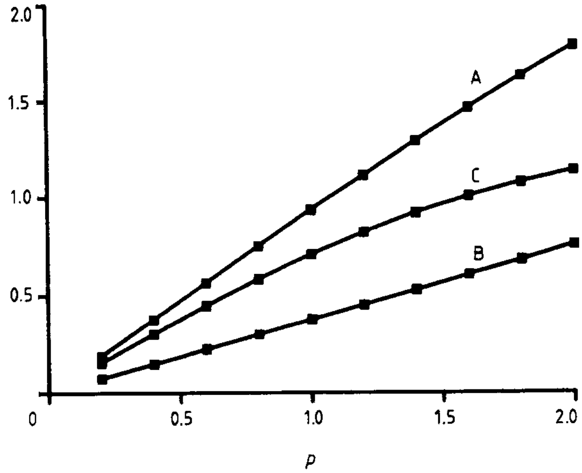

Figure 1. The curves $\alpha(p)$ for the spectral flow on a 512^2 grid (A) and for the lattice gas flow mapped onto a 128^2 grid (C) and a 512^2 grid (B).

the bending of curve C is due to poor statistics has to be ruled out (we are indebted to one of the referees for focusing our attention on this point).

A possible explanation of the deviation of curve C from a linear behaviour can be attempted in terms of the concept of the multifractal introduced by Frisch and Parisi [9]. These authors show that for a two-dimensional flow containing several singular scales one can write

$$\alpha(p) = \min_h(ph + 2 - d(h)) \tag{2}$$

where $d(h)$ is the fractal dimension of the set S_h of the points such that $\Delta v(r) \equiv v(x+r) - v(x) \sim r^h$. The above equation results from a saddle-point estimate of the quantity $\langle \Delta v^p \rangle \sim \int \mathrm{d}\mu(h) r^{|ph+2-d(h)|} \sim r^{\alpha(p)}$ where $\mu(h)$ is a measure concentrated in the region where $d(h) > 0$ and $r^{2-d(h)}$ is the probability for a point to belong to the set S_h (for more details see Benzi *et al* [6]). From the equation (2) one can interpret curve C of figure 1 by assuming $\min_h(2 - d(h)) = 0$, that is, a multifractal set with a fractal dimension of two. However, this is a kind of 'special' multifractal which results from the application of the averaging procedure to a field containing a significant amount of noise (we will come to this point again later on). In fact, for small values of p most of the statistical weight in the averaging procedure is given to the regular part of the field while for high p the role of the fluctuations becomes more and more relevant. Therefore, when filtering the noise out (which is precisely what we do by averaging) it is reasonable to expect that for small p the curve B tends 'quicker' to the analytical form than it does at high p, thereby causing the non-linear behaviour displayed by curve C. This is consistent with the quadratic best fit $\alpha = 0.84p - 0.11p^2$ we obtained for curve C.

In any event, it is important to stress that as far as the main focus of this study is concerned, namely the limit of $\alpha(p)$ as $p \to 0$, all the cases examined yielded the same result, i.e. $\alpha(p \to 0) = 0$ within two digits. Hence, we conclude that the gradient of the velocity field is singular but the set of the points where this singularity is concentrated has a fractal dimension of two, i.e. no fractal structure has emerged.

To interpret this result, let us recall that the non-analyticity of the lattice gas field is related to the considerable amount of noisy energy contained in the short-scale component of the spectrum (see figure 2). This noisy energy is in turn a manifestation of the fact that the paths traced by the pseudo-particles ('Boolean molecules') in the microscopic lattice are continuous but not differentiable. In other words, the pseudo-particles undergo a sort of random walk whose irregularities become more and more apparent as the spacing of the averaging grid is down-sized. It is therefore natural to represent the velocity field as the sum of a smooth component V and a fluctuating component η which results from the presence of the molecular noise. Hence, we write the velocity v as

$$v = V + \eta \tag{3}$$

where the vector notation has been relaxed for the sake of simplicity. The amplitude of the fluctuations decreases with the square root of the number of bits/cell available to build up a single averaged velocity. This can be seen either by regarding v in the cell as the stochastic variable defined by the sum of n^2 Boolean variables, or by regarding the noise η as the fast component of the field associated with the high-frequency part of the k spectrum. By averaging over a cell of size n (in units of the lattice gas spacing), one sets an outer cutoff $k_n = 2\pi/n$ to the spectrum which regularises the behaviour of the field in real space. As a result, the structure functions can be written as:

$$A_p(r) = \langle|\delta V(r) + \delta\eta(r)|^p\rangle \tag{4}$$

where δ refers to the variation between two points a distance r apart.

Since V is smooth, δV scales like Ar where A is proportional to the average gradient of the fluid flow. On the other hand $\delta\eta$ scales like

$$\delta\eta = C\gamma(r) \tag{5}$$

where $C \sim n^{-1}$ and γ is an unknown function which depends on the structure of the noise.

The behaviour of the functions A_p is the result of the competition between the regular component V, which tends to produce $\alpha = p$, and the noisy component. If we assimilate the noise to a sort of fractional random walk [10], it is legitimate to assume

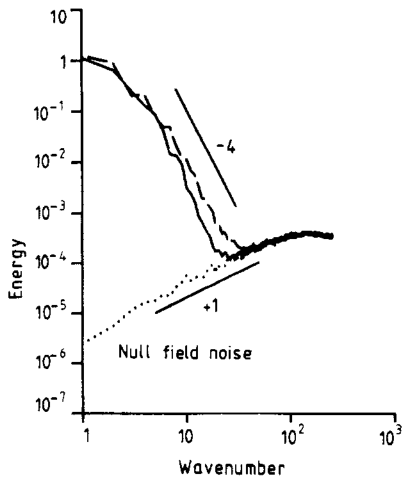

Figure 2. Energy spectrum $E(k)$ of the automaton flow at the beginning (broken curve) and at the end (full curve) of the simulation. The dotted curve is the stochastic representation of the null velocity field.

that $\gamma \sim r^c$ with c some exponent between zero and one. In this case the noise would tend to produce a law $\alpha \sim cp$ so that no possibility other than $\alpha(p \to 0) \to 0$, i.e. no fractal formation, would be left. On the other hand, at a macroscopic level the noise seems to give rise to a sort of fragmentation of the vorticity contours (see figure 3) which evokes a certain analogy with the eddy-fragmentation process assumed by the β model and its variants.

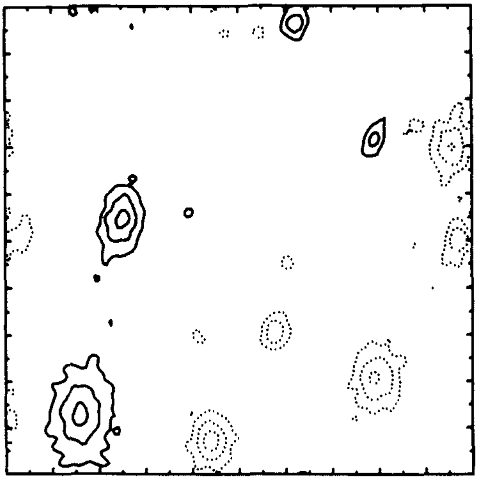

Figure 3. 'Coastlines' of vorticity of the automaton at the end of the simulation. The figure is obtained by retaining only the smallest (wavenumber $\leqslant 32$) components of the harmonic analysis of the Boolean field. The contours are 2.5, 7.5, 15 (full curves) and -2.5, -7.5, -15 (dotted curves).

The data shown in figure 1 indicate that even if this fragmentation process takes place, it is nonetheless space filling and consequently no fractal set can appear.

In view of the preceding considerations we are led to the following picture of dissipation in the lattice gas. The energy of the macroscopic organised motion (vortices) is converted into chaotic motion ('heat') via interparticle collisions. The fragmentation process associated with this chaotic motion is, however, space filling and reflects the regularity of the large-scale configuration which promotes it. As a result, the emergence of fractal structures is inhibited and dissipation is still a homogeneous non-intermittent process like in a noiseless fluid theory.

This provides a further non-trivial analogy between the behaviour of a lattice gas and the Navier–Stokes picture of a fluid flow.

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